Collaborative Learning in Mathematics

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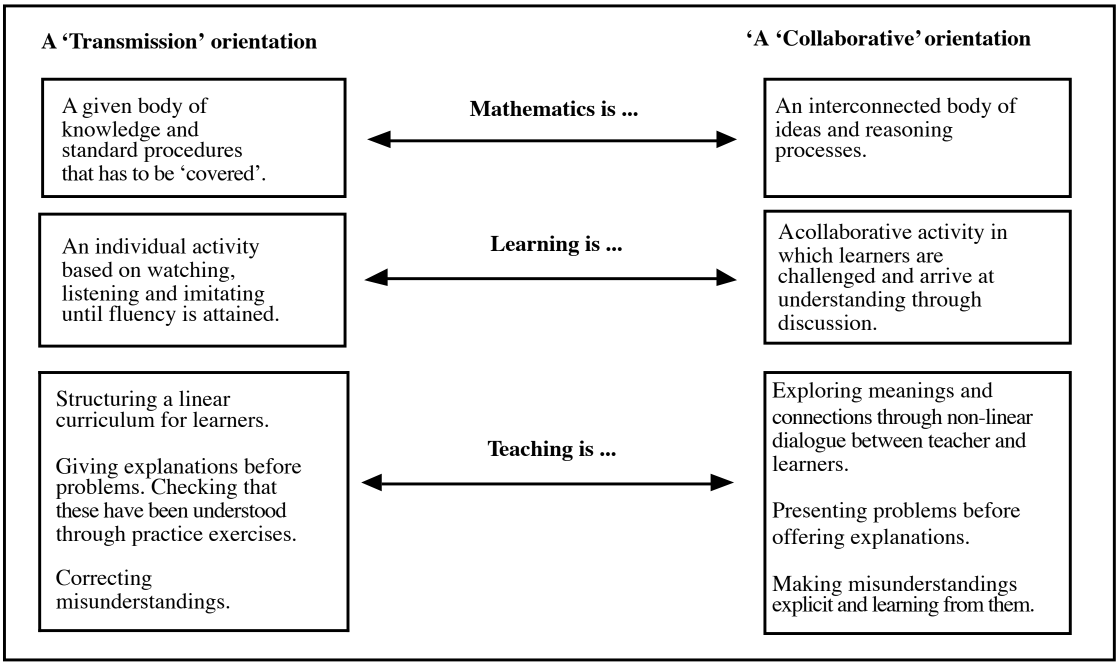
Introduction

Since 1979, I have conducted research and development with my colleagues at the University of Nottingham into more effective ways of teaching and learning mathematical concepts and strategies. This work is now coming to fruition through the dissemination of professional development resources: *Improving Learning in Mathematics* (Swan, 2005), *Thinking through Mathematics* (Swain & Swan, 2007; Swan & Wall, 2007) and *Bowland Maths* (Swan, 2008). These multimedia resources are now being distributed to schools and colleges across England.

These projects have similar aims. The first aim is to help students to adopt more active approaches towards learning. Our own research shows that many students view mathematics as a series of unrelated procedures and techniques that have to be committed to memory. Instead, we want them to engage in discussing and explaining ideas, challenging and teaching one another, creating and solving each other's questions and working collaboratively to share methods and results. The second aim is to develop more challenging, connected, collaborative orientations towards their teaching (Swan, 2005):



Traditional, 'transmission' methods in which explanations, examples and exercises dominate do not promote robust, transferrable learning that endures over time or that may be used in non-routine situations. They also demotivate students and undermine confidence. In contrast, the model of teaching we have adopted emphasises the interconnected nature of the subject and it confronts common conceptual difficulties through discussion. We also reverse traditional practices by allowing students opportunities to tackle problems before offering them guidance and support. This encourages them to apply pre-existing knowledge and allows us to assess and then help them build on that knowledge. This approach has a thorough empirically tested research base (Swan, 2006). The main principles are summarised below.

Teaching is more effective when it ...

• **builds on the knowledge students already have;**	This means developing formative assessment techniques and adapting our teaching to accommodate individual learning needs (Black & Wiliam, 1998).
• **exposes and discusses common misconceptions**	Learning activities should exposing current thinking, create 'tensions' by confronting students with inconsistencies, and allow opportunities for resolution through discussion (Askew & Wiliam, 1995).
• **uses higher-order questions**	Questioning is more effective when it promotes explanation, application and synthesis rather than mere recall (Askew & Wiliam, 1995).
• **uses cooperative small group work**	Activities are more effective when they encourage critical, constructive discussion, rather than argumentation or uncritical acceptance (Mercer, 2000). Shared goals and group accountability are important (Askew & Wiliam, 1995).
• **encourages reasoning rather than 'answer getting'**	Often, students are more concerned with what they have 'done' than with what they have learned. It is better to aim for depth than for superficial 'coverage'.
• **uses rich, collaborative tasks**	The tasks we use should be accessible, extendable, encourage decision-making, promote discussion, encourage creativity, encourage 'what if' and 'what if not?' questions (Ahmed, 1987).
• **creates connections between topics**	Students often find it difficult to generalise and transfer their learning to other topics and contexts. Related concepts (such as division, fraction and ratio) remain unconnected. Effective teachers build bridges between ideas (Askew et al., 1997).
• **uses technology in appropriate ways**	Computers and interactive whiteboards allow us to present concepts in visual dynamic and exciting ways that motivate students.

Designing teaching activities

Such principles are easy to state, but 'engineering' them so that they work in practice is very difficult. We have worked with teachers to develop and trial activity-based sessions that exemplify the above principles. These sessions include resources for students, full teaching notes, software (where necessary) and film clips of teachers 'in action'. The activities may be categorised into five 'types' that encourage distinct ways of thinking and learning:

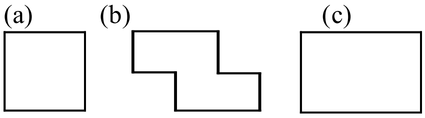
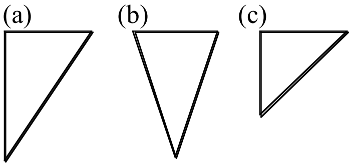
Classifying mathematical objects	Students devise their own classifications for mathematical objects, and/or apply classifications devised by others. In doing this, they learn to discriminate carefully and recognise the properties of objects. They also develop mathematical language and definitions. The objects might be anything from shapes to quadratic equations.
Interpreting multiple representations	Students work together matching cards that show different representations of the same mathematical idea. They draw links between representations and develop new mental images for concepts.
Evaluating mathematical statements	Students decide whether given statements are always, sometimes or never true. They are encouraged to develop mathematical arguments and justifications, and devise examples and counterexamples to defend their reasoning. For example, is the following statement always, sometimes or never true? If sometimes, then when? "Jim got a 15% pay rise. Jane got a 10% pay rise. So Jim's pay rise was greater than Jane's."

Creating problems	Students are asked to ceate problems for other students to solve. When the ‘solver’ becomes stuck, the problem ‘creators’ take on the role of teacher and explainer. In these activities, the ‘doing’ and ‘undoing’ processes of mathematics are exemplified. For example, one partner may create an equation, then the other tries to solve it.
Analysing reasoning and solutions	Students compare different methods for doing a problem, organise solutions and/ or diagnose the causes of errors in solutions. They begin to recognise that there are alternative pathways through a problem, and develop their own chains of reasoning.

1. Classifying mathematical objects

In these activities, students devise their own classifications for mathematical objects, and/or apply classifications devised by others. In doing this, they learn to discriminate carefully and recognise the properties of objects. They also develop mathematical language and definitions. The objects might be anything from geometric shapes to quadratic equations.

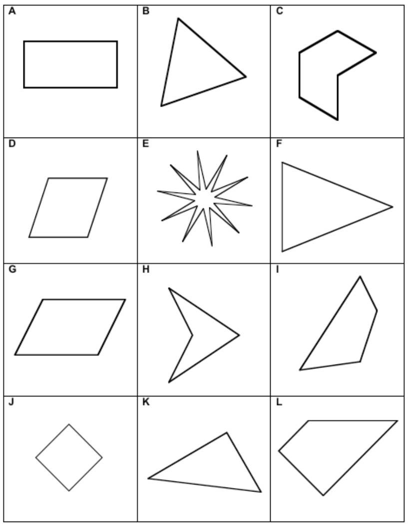
Perhaps the simplest form of classification activity is to examine a set of three objects and identify, in turn, why each one might be considered the ‘odd one out’. For example, in the triplets below, how can you justify each of (a), (b), (c) as the odd one out? Each time, try to produce a new example to match the ‘odd one out’.

(a) (b) (c) 	(a) sin 60° (b) cos 60° (c) tan 60°
(a) a fraction (b) a decimal (c) a percentage	(a) $y = x^2-6x+8$ (b) $y = x^2-6x+9$ (c) $y = x^2-6x+10$
(a) (b) (c) 	(a) 20, 14, 8, 2, ... (b) 3, 7, 11, 15, ... (c) 4, 8, 16, 32, ...

For example, in the first example, (a) may be considered the odd one as it has a different perimeter to the others, (b) may be considered the odd one because it is not a rectangle and (c) may be considered the odd one because it has a different area to the others. Each time a reason is given, students identify different properties of the objects.

Students may also be asked to sort a large collection of cards containing mathematical 'objects' into two sets according to criteria of their own choice. They then subdivide each set into two subsets using further criteria. They might then generate further objects for each set. Through sharing criteria, mathematical language and definitions are developed. Students may then be given two-way grids on which they can classify cards. Where they find that one cell of the grid is empty, they try to find an example that will fit, otherwise they try to explain why it is impossible. Some examples of the cards and grids are given below.

Shape



	No rotational symmetry	Rotational symmetry
No lines of symmetry		
One or two lines of symmetry		
More than two lines of symmetry		

Quadratic functions

$y = x^2 + 2x + 4$	$y = x^2 - 5x + 4$
$y = 2x^2 - 5x - 3$	$y = x^2 - 4x + 4$
$y = x^2 + 7x - 3$	$y = 4 + 3x - x^2$
$y = x^2 + 5x - 2$	$y = 6x - x^2 - 9$
$y = x^2 - 3x - 1$	$y = x^2 + 10x + 9$
$y = x^2 + x + 3$	$y = x^2 + 4x + 4$
$y = x^2 - 2\sqrt{3}x + 3$	$y = 3x - x^2 + 7$

	Factorises with integers	Does not factorise with integers
Two *x* intercepts		
No *x* intercepts		
Two equal *x* intercepts		
Has a minimum point		
Has a maximum point		
y intercept is 4.		

2. Interpreting Multiple Representations

Mathematical concepts have many representations; words, diagrams, algebraic symbols, tables, graphs and so on. These activities are intended to allow these representations to be shared, interpreted, compared and grouped in ways that allow students to construct meanings and links between the underlying concepts.

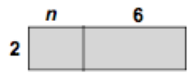
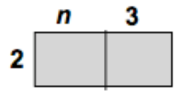
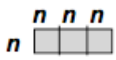
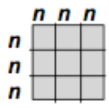
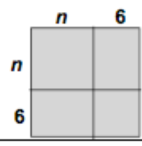
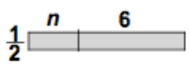
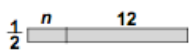
In most classrooms, a great deal of time is already spent on the technical skills needed to construct and manipulate representations. These include, for example, adding numbers, drawing graphs and manipulating formulae. While technical skills are necessary and important, this diet of practice must be balanced with activities that offer students opportunities to reflect on their meaning. These activities provide this balance. Students focus on interpreting rather than producing representations.

Perhaps the most basic and familiar activities in this category are those that require students to match pairs of mathematical objects if they have an equivalent meaning. This may be done using domino-like activities. More complex activities may involve matching three or more representations of the same object. Typical examples might involve matching:

- Times and measures expressed in various forms (e.g. 24-hour clock times and 12-hour clock times);
- Number operations (e.g. notations for division)
- Numbers and diagrams (e.g. Decimals, fractions, number lines, areas);
- Algebraic expressions (e.g. words, symbols, area diagrams – see below);
- Statistical diagrams (e.g. Frequency tables, Cumulative frequency curves).

The discussion of misconceptions is also encouraged if carefully designed distracters are also included. The example below shows one possible set of cards for matching. The sets of cards used in the sessions also contain blank cards so that students are not able to complete them using elimination strategies. Students are asked to construct the missing cards for themselves.

Interpreting algebraic notation

$\frac{n+6}{2}$	$3n^2$	Square *n*, then multiply by three	*n*: 1, 2, 3, 4 *Ans*: 14, 16, 18, 20	
$2n+12$	$2n+6$	Add six to *n*, then multiply by two.	*n*: 1, 2, 3, 4 *Ans*: , , 81, 144	
$2(n+3)$	$\frac{n}{2}+6$	Add six to *n*, then divide by two	*n*: 1, 2, 3, 4 *Ans*: , 10, 15, 22	
$(3n)^2$	$(n+6)^2$	Divide *n* by two, then add three	*n*: 1, 2, 3, 4 *Ans*: 3, , 27, 48	
$n^2+12n+36$	$\frac{n}{2}+3$	Add six to *n*, then square the answer	*n*: 1, 2, 3, 4 *Ans*: , , 81, 100	
n^2+6	Add three to *n* then multiply by two.	Square *n*, then multiply by nine	*n*: 1, 2, 3, 4 *Ans*: , 4, , 5	
n^2+6^2	Multiply *n* by two then add twelve	Multiply *n* by two, then add six.	*n*: 1, 2, 3, 4 *Ans*: 6.5, 7, 7.5, 8	

When using such card matching activities, we have found that students often begin quickly and superficially, making many mistakes in the process. Some become 'passengers' and let others do all the work. The teacher's role is therefore to ensure that students:

- take their time and do not rush through the task,
- take turns at matching cards, so that everyone participates;
- explain their reasoning and write reasons down;
- challenge each other when they disagree;
- find alternative ways to check answers (e.g. using calculators, finding areas in different ways, manipulating the functions);
- create further cards to show what they have learned.

Often, students like to stick their cards down onto a poster and write their reasoning around the cards. For example, they might write down how they know that $9n^2$ and $(3n)^2$ correspond to the same area in the cards shown above. It is important to give all students an equal opportunity to develop their written reasoning skills in this way. If a group does not share the written work out equally, additional opportunities for written reasoning need to be created, perhaps through short, individual assignments.

These card sets are powerful ways of encouraging students to see mathematical ideas from a variety of perspectives and to link ideas together.

3. Evaluating mathematical statements

These activities offer students a number of mathematical statements or generalisations. Students are asked to decide whether the statements are always, sometimes or never true and give explanations for their decisions. Explanations usually involve generating examples and counterexamples to support or refute the statements. In addition, students may be invited to add conditions or otherwise revise the statements so that they become 'always true'.

This type of activity develops students' capacity to explain, convince and prove. The statements themselves can be couched in ways that force students to confront some common difficulties and misconceptions. Statements might be devised at any level of difficulty. They might concern, for example:

- the size of numbers ('numbers with more digits are greater in value');
- number operations ('multiplying makes numbers bigger');
- area and perimeter ('shapes with larger areas have larger perimeters');
- algebraic generalisations ('$2(n+3) = 2n+3$');
- enlargement ('if you double the lengths of the sides, you double the area');
- sequences ('if the sequence of terms tends to zero, the series converges');
- calculus ('continuous graphs are differentiable').

... and so on.

On the next page are some examples. In each case, (except for the probability example), the statements may be classified as 'always, sometimes or never' true.

Students may enjoy working together arguing about the statements and showing their agreed reasoning on posters.

Throughout this process, the teacher's role is to:

- encourage students to think more deeply, by suggesting that they try further examples. ('Is this one still true for decimals or negative numbers?'; 'What about when I take a bite out of a sandwich? How does that change the perimeter and area?')
- challenge students to provide more convincing reasons. ('I can see a flaw in that argument'. 'What happens when ....?')
- play 'devil's advocate'. ('I think this is true because... Can you convince me I am wrong?')

Sample cards for discussion: Always, Sometimes or Never true?

Digits Numbers with more digits are greater in value	**Add a nought** To multiply by ten, you just add nought on the right hand end of the number.
Pay rise Max gets a pay rise of 30%. Jim gets a pay rise of 25%. So Max gets the bigger pay rise.	**Sale** In a sale, every price was reduced by 25%. After the sale every price was increased by 25%. So prices went back to where they started.
Area and perimeter When you cut a piece off a shape you reduce its area and perimeter.	**Right angles** A pentagon has fewer right angles than a rectangle.
Birthdays In a class of ten students, the probability of two students being born on the same day of the week is one.	**Lottery** In a lottery, the six numbers 3, 12, 26, 37, 44, 45 are more likely to come up than the six numbers 1, 2, 3, 4, 5, 6.
Bigger fractions If you add the same number to the top and bottom of a fraction, the fraction gets bigger in value.	**Smaller fractions** If you divide the top and bottom of a fraction by the same number, the fraction gets smaller in value.
Square roots The square root of a number is less than or equal to the number	**Consecutive numbers** If you add n consecutive numbers together, the result is divisible by n.

4. *Creating Problems*

In this type of activity, students are given the task of devising their own mathematical problems. They try to devise problems that are both challenging and that they know they can solve correctly. Students first solve their own problems and then challenge other students to solve them. During this process, they offer support and act as 'teachers' when the problem solver becomes stuck.

Students may be asked to construct their own problems for a variety of reasons. These include:

- Enabling students to reflect on their own capabilities (e.g., "Make up some problems that test all the ways in which one might use Pythagoras' theorem").
- Promoting an awareness of the range of problem types that are possible.
- Focusing attention on the various features of a problem that influence its difficulty (e.g., size of numbers, structure, context).
- Encouraging students to consider appropriate contexts in which the mathematics may be used (e.g. create a range of problems about directed numbers using a money context).
- Helping students to gain 'ownership' over their mathematics and confidence when explaining to others.

At its most basic, this strategy may follow on from any exercise that the students have been engaged in; " You've been working on these questions, now make up some more of your own for a neighbour to solve." In or materials, however, creating problems has a more central role to play. The activities are mainly of two types (see examples on facing page).

(i) Exploring the *doing* and *undoing* processes in mathematics.

In these situations, the poser creates a problem using one process, then the solver attempts to reverse that process in order to find a solution. In some cases the solution may not be the one expected, and this can create some useful discussion. In most situations, the poser has an easier task than the solver. This ensures that the task is solvable.

(ii) Creating variants of existing questions.

It is helpful to do this in stages. Firstly, presented with a given question, ask 'what other questions may have been asked?' This helps students to explore the structure of the situation more fully. Secondly, the student tries to change the question in small ways. The numbers might be changed, for example. What numbers make a solution impossible? The diagram might be altered, and so on. Instead of just doing one question, the student becomes aware that this question is just one example of a class of problems that might have been asked.

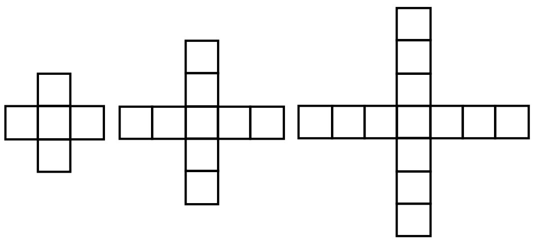
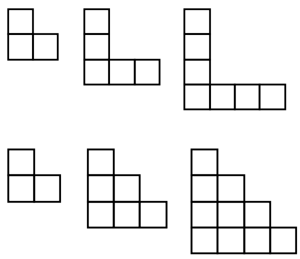
Throughout the process, the teacher's role is to:

- explain and support the process of problem creation;
- encourage students to support each other in solving the questions;
- challenge students to explain why some problems appear to have several alternative solutions.

Exploring the doing and undoing processes in mathematics

Doing: The problem poser..	**Undoing: The problem solver..**
• calculates the area and perimeter of a rectangle (e.g. 5 cm x 7cm).	• finds a rectangle with the given area (35 cm) and perimeter (24 cm).
• writes down an equation of the form *y=mx+c* and plots a graph.	• tries to find an equation that fits the resulting graph.
• expands an expression such as (x+3)(x-2)	• factorises the resulting expression: x^2+x-6
• generates an equation step-by-step, starting with x = 4 and 'doing the same to both sides': times 10 $10x = 40$ Add 9 $10x+9 = 49$ Divide by 8 $\frac{10x+9}{8} = 6.125$ take 7 $\frac{10x+9}{8} - 7 = -0.875$	• solves the resulting equation: $\frac{10x+9}{8} - 7 = -0.875$
• writes down a polynomial and differentiates it: $x^5 + 3x^2 - 5x + 2$	• integrates the resulting function: $5x^4 + 6x - 5$
• writes down five numbers 2, 6, 7, 11, 14 and finds their mean, median, range	• tries to find five numbers with the resulting values of mean=8; median=7 and range=12.

Creating variants of existing questions.

Original exam question	**Possible revisions**
Some cross patterns are made of squares.  Diagram 1 5 squares Diagram 2 9 squares Diagram 3 13 squares (a) How many squares will be in diagram 6? (b) Write down an expression for the number of squares in diagram *n*. (c) Which diagram will have 125 squares?	**Write new questions for the original situation:** • Can you have a diagram with 500 squares? How can you be sure? • The first cross is 3 squares long. How long is the nth cross? • The first diagram has a perimeter of 12, what is the perimeter of the 4th diagram? The 100th diagram? The nth diagram? • Is it possible to draw a cross diagram with a perimeter of 100? How can you be sure? **Or, change the original situation:** 

5. *Analysing reasoning and solutions*

The activities suggested here are designed to shift the predominant emphasis from 'getting the answer' towards a situation where students are able to evaluate and compare different forms of reasoning.

(i) *Comparing different solution strategies*

In many mathematics lessons, students apply a single taught method to a variety of questions. It is comparatively rare to find lessons that aim to compare a range of methods for tackling a few problems. Many students are left feeling that if they do not know 'the right method' then they cannot even begin to attempt a problem. Others are stuck with methods that, while generating correct answers, are inefficient and inflexible. These activities are designed to allow students to compare and discuss alternative solution strategies to problems, thus increasing their confidence and flexibility in using mathematics. When 'stuck', they become more inclined to 'have a go' and try something. They thus become more powerful problem solvers. In the following example, students are asked to find as many different ways as they can of solving a simple proportion problem.

Paint prices  1 litre of paint costs £15. What does 0.6 litres cost?	Chris: It is just over a half, so it would be about £8. Sam: I would divide 15 by 0.6. You want a smaller answer. Rani: I would say one fifth of a litre is £3, so 0.6 litres will be three times as much, so £9. Tim: I would multiply 15 by 0.6. Teacher: Do your methods give the same answers? If I change the 0.6 to a different number, say, 2.6, would your methods change? Why or why not? Does the method depend on the numbers?

(ii) *Correcting mistakes in reasoning*

These activities require students to examine a complete solution and identify and correct errors. The activity may also invite the student to write advice to the person who made the error. This puts the student in a critical, advisory role. Often the errors that are exhibited are symptomatic of common misconceptions. In correcting these, therefore, students have to confront and comment on alternative ways of thinking. In the example below, four students Harriet, Andy, Sara and Dan are discussing a common percentages misconception.

Rail prices

In January, fares went up by 20%.
In August, they went down by 20%.

Sue claims that:
"The fares are now back to what they were before the January increase".

Do you agree?
If not, what has she done wrong?

Harriet: That's wrong, because...they went up by 20%, say you had £100 that's 5, no 10.
Andy: Yes, £10 so its 90 quid, no 20% so that's £80. 20% of 100 is 80,... no 20.
Harriet: Five twenties are in a hundred.
Dan: Say the fare was 100 and it went up by 20%, that's 120.
Sara: Then it went back down, so that's the same.
Harriet: No, because 20% of 120 is more than 20% of 100. It will go down by more so it will be less. Are you with me?
Andy: Would it go down by more?
Harriet: Yes because 20% of 120 is more than 20% of 100.
Andy: What is 20% of 120?
Dan: 96...
Harriet: It will go down more so it will be less than 100.
Dan: It will go to 96.

(iii) Putting reasoning in order

Students often find it difficult to produce an extended chain of reasoning. This process may be helped (or 'scaffolded') by offering the steps in the reasoning on cards, and then asking students to correctly sequence the steps of the solution or argument. The focus of attention is thus on the underlying logic and structure of the solution rather than on its technical accuracy.

Cut up the following cards. Rearrange them to form two proofs.

The first should prove that: **If n is an odd number, then n^2 is an odd number**

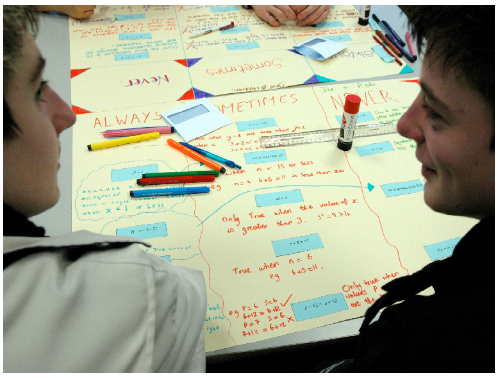
The second should prove that: **If n^2 is an odd number, then n is an odd number.**
You may not need to use all the cards.

If n is odd	**So n is odd**
$n = 2m + 1$ **for some integer m**	$= 2k$ **where $k = 2m^2$**
$(2m + 1)^2 = 4m^2 + 4m + 1$	**But n^2 is odd**
$(2m)^2 = 4m^2$	**So n^2 is odd**
If n is even	$n = 2m$ **for some integer m**
So n^2 is even	$= 2k + 1$ **where $k = 2m(m + 1)$**
If n^2 is odd	$n^2 = 2m + 1$ **for some integer m**

Resources for learning

In addition to the activities, we also considered generic ways in which resources such as posters, mini-whiteboards, and computer software may be used to enhance the quality of learning.

Posters are often used in schools and colleges to display the finished, polished work of students. In our work, however, we use them to promote collaborative thinking. The posters are not produced *at the end* of the learning activity; they *are* the learning activity and they show all the thinking that is taking place. We often ask students to solve a problem in two different ways on the poster and then display the results for other students to comment on.



Mini-whiteboards have rapidly become an indispensable aid to whole class discussion for several reasons:

- During whole class discussion, they allow the teacher to ask new *kinds* of question (typically beginning: 'Show me....').
- When students hold their ideas up to the teacher, it is possible to see at a glance what *every* student thinks.
- They allow students to, simultaneously, present a range of written and/or drawn responses to the teacher and to each other.
- They encourage students to use private, rough working that may be quickly erased.

Examples of a range of ‘Show me..’ questions are given below. Notice that most of these are ‘open questions’ that allow a range of responses. It is worth encouraging a range of such responses with instructions like: ‘Show me a really *different* example’; ‘Show me a complicated example’; ‘Show me an example that is different from everyone else on your table” (Watson & Mason, 1998).

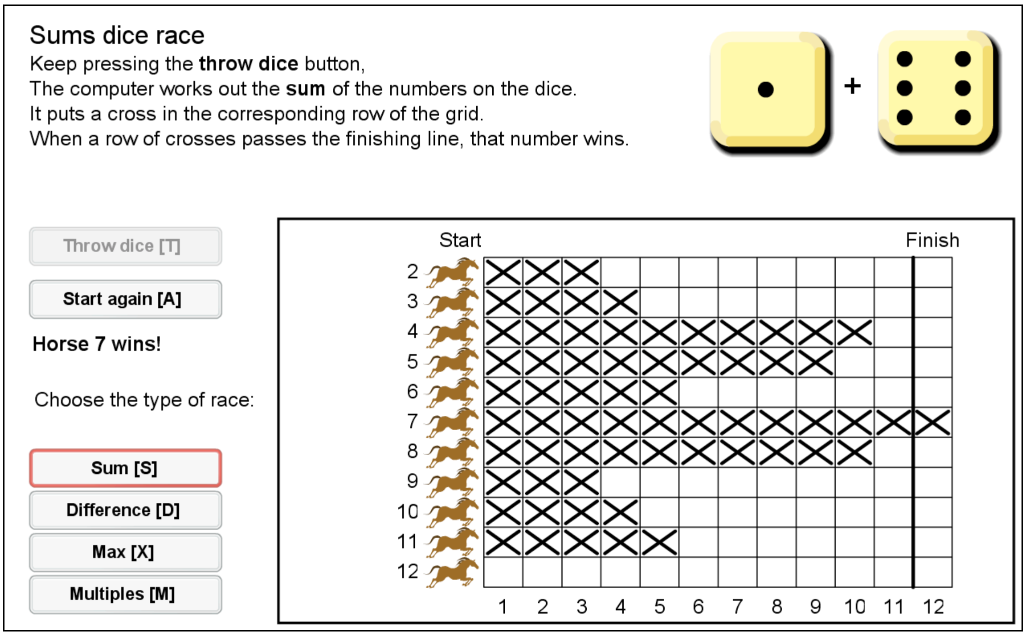
Typical ‘show me’ open questions.

Show me:

- Two fractions that add to 1 ..... Now show me a different pair.
- A number between 0.6 and 0.7....Now between 0.6 and 0.61.
- A number between $\frac{1}{3}$ and $\frac{1}{4}$.... Now between $\frac{1}{3}$ and $\frac{2}{7}$.
- The name of something that weighs about 1 kg. ..... 0.1 kg?
- A hexagon with two reflex angles ... A pentagon with four right angles.
- A shape with an area of 12 square units …and a perimeter of 16 units.
- A set of 5 numbers with a range of 6 … and a mean of 10

Computer software has been selected and developed to enhance the teaching of some of the more difficult concepts. These include programs that are designed to help students interpret a range of representations, such as functions, graphs and statistical charts; situations to explore, such as a 3-dimensional ‘building’ activity, aimed at developing facility with plans and elevations; and some individual practice programs, developing fluency when creating and solving equations.

For example, the program *Dice races* encourages students to make statistical predictions, carry out experiments, generate data and explain the patterns observed.



What do the teachers and students think?

The response from the teachers and students has been very encouraging. Some have been challenged to reconsider long-held assumptions about teaching and learning. The following quotes are fairly typical.

> Lessons are now far more enjoyable for students. I would like to adapt the materials for all teaching sessions.
>
> I always asked a lot of questions and thought they were really helpful. I now realise these sometimes closed discussion down or cut them off. Now I step back and let the discussion flow more. This is very hard to do.
>
> Most students have been positive with the increasing participation in discussion, creating questions etc. some found some of the activities difficult - but this challenged them and they got a lot out of it. Most have gained confidence in articulating ideas. Some are still afraid to make mistakes however.
>
> The good thing about this was, instead of like working out of your textbook, you had to *use* your brain before you could go anywhere else with it. You had to actually sit down and think about it. And when you *did* think about it you had someone else to help you along if you couldn't figure it out for yourself, so if they understood it and you didn't they would help you out with it. If you were doing it out of a textbook you wouldn't get that help. (Lauren, student aged 16)

The response to these materials from schools has been extremely positive. In May 2006, during their regular inspections of post-16 education institutions, Ofsted began to come across these resources in use (Ofsted, 2006). As they noted in their report:

> These materials encouraged teachers to be more reflective and offered strategies to encourage students to think more independently. They encouraged discussion and active learning in AS, A level and GCSE lessons. (para 32).
>
> While some colleges were just dipping into the resources, a few had used the full package to transform teaching and learning across an entire mathematics team. (para 33).

There is now a growing body of evidence that well-designed tasks, and supporting resources that illustrate these tasks in use, can contribute to the transformation of teaching and learning.

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